# Cospectrality, latent symmetries and isospectral reductions

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Workshop on local symmetries in wave physics
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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

# Outline

- Motivation: Discrete systems and graphs
- Local symmetries in graphs
- Isospectral reductions
- Application of these insights to the problem of pretty good state transfer

Main goal of this talk: Convincing you that we can profit a lot from graph theory!

# Motivation: Discrete systems and graphs

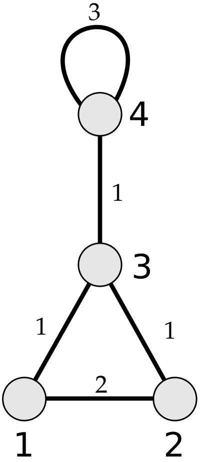
The topic of this talk are systems described by matrices, such as:

- Tight-binding systems: Spin networks, evanescently coupled waveguides, acoustic airchannels[1]
- Multi-level atoms (think about STIRAP).
- Graphs: Transportation networks, food webs, social networks, power grids, the world wide web, citation networks, gene regulatory networks, chemical reaction networks.

The matrices describing graphs are often real-valued and symmetric; I will restrict myself to such matrices in the rest of this talk.

[1] Zheng et al., "Observation of Edge Waves in a Two-Dimensional Su-Schrieffer-Heeger Acoustic Network".

# An example graph and its adjacency matrix




$$M = \begin{pmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

# Typical problems in graph theory

- How well are the nodes interconnected? How many edges may one remove before the graph splits into disconnected pieces?
- How can I get from node $A$ to node $B$? Applications in computer networks, as well as in all navigation systems.
- Which nodes are “more important” (fuzzy definition!) than others? An example for this is the Google search algorithm.

# "Local" symmetries in graphs

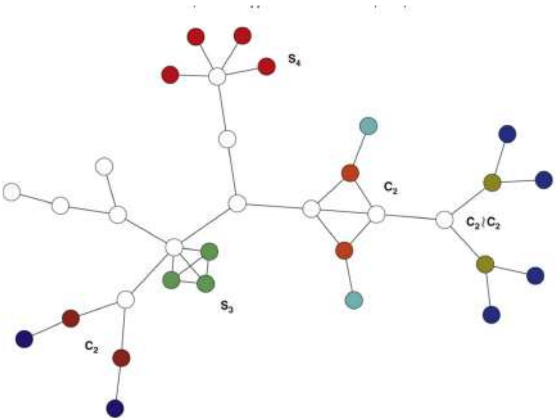
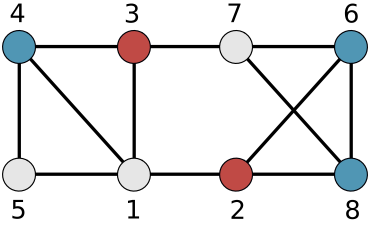



Figure: Taken from[2]

[2] MacArthur, Sánchez-García, and Anderson, "Symmetry in Complex Networks".

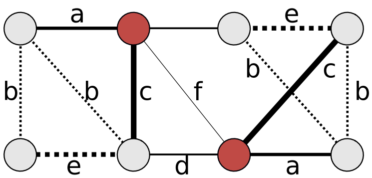
# Truly local symmetries in graphs

Consider the following graph:



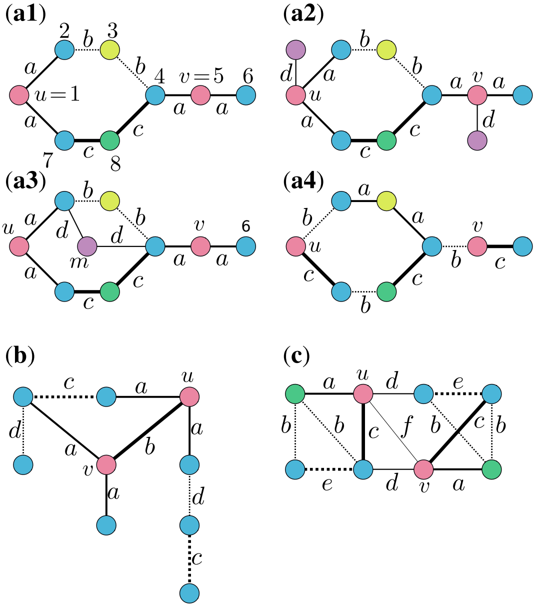

$$\begin{pmatrix}
0.415 & 0.0877 & -0.381 & 0.692 & 0 & -0.365 & 0.218 & -0.127 \\
0.438 & -0.505 & 0.153 & -0.343 & 0 & 0 & 0.471 & -0.437 \\
0.438 & 0.505 & -0.153 & -0.343 & 0 & 0 & -0.471 & -0.437 \\
0.315 & -0.292 & -0.430 & 0.0959 & 0 & 0.73 & -0.174 & 0.241 \\
0.216 & 0.0877 & -0.381 & -0.500 & 0 & -0.365 & 0.218 & 0.609 \\
0.315 & 0.292 & 0.430 & 0.0959 & -0.707 & 0.183 & 0.174 & 0.241 \\
0.315 & -0.467 & 0.332 & 0.0959 & 0 & -0.365 & -0.609 & 0.241 \\
0.315 & 0.292 & 0.430 & 0.0959 & 0.707 & 0.183 & 0.174 & 0.241
\end{pmatrix}$$

# Local symmetries in graphs II




All eigenvectors can be chosen to be locally symmetric on the two red vertices, no matter how the parameters are chosen.

# Local symmetries in graphs III



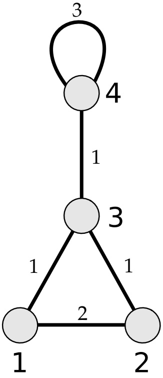
# Cospectrality

In graph theory (but more general, in linear Algebra!), a matrix $M$ is said to have cospectral indices $u, v$ (vulgo: $u, v$ are cospectral) if any of the following equivalent conditions apply[3]

- $u$ and $v$ are cospectral.
- The spectrum of $M \setminus u$ is equal to the spectrum of $M \setminus v$.
- All eigenstates can be chosen to have parity $\pm 1$ with respect to an exchange of $u$ and $v$.
- $(M^k)_{u,u} = (M^k)_{v,v}$ for all integer $k > 0$.

[3] Eisenberg, Kempton, and Lippner, "Pretty Good Quantum State Transfer in Asymmetric Graphs via Potential".

# The meaning of (positive integer) matrix powers




$$M = \begin{pmatrix} 0 & \boxed{2} & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{pmatrix} \qquad \boxed{① \xrightarrow{2} ②} = \boxed{2}$$

$$M^2 = \begin{pmatrix} \boxed{5} & 1 & 2 & 1 \\ 1 & 5 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ 1 & 1 & 3 & 10 \end{pmatrix} \qquad \boxed{\begin{matrix} ① \xrightarrow{2} ② \xrightarrow{2} ① \\ + \\ ① \xrightarrow{1} ③ \xrightarrow{1} ① \end{matrix}} = \boxed{\begin{matrix} 2 \cdot 2 \\ + \\ 1 \cdot 1 \end{matrix}}$$

$$M^3 = \begin{pmatrix} 4 & \boxed{12} & 7 & 5 \\ 12 & 4 & 7 & 5 \\ 7 & 7 & 7 & 12 \\ 5 & 5 & 12 & 33 \end{pmatrix} \qquad \boxed{\begin{matrix} ① \xrightarrow{2} ② \xrightarrow{2} ① \xrightarrow{2} ② \\ + \\ ① \xrightarrow{1} ③ \xrightarrow{1} ① \xrightarrow{2} ② \\ + \\ ① \xrightarrow{2} ② \xrightarrow{1} ③ \xrightarrow{1} ② \end{matrix}} = \boxed{\begin{matrix} 2 \cdot 2 \cdot 2 \\ + \\ 1 \cdot 1 \cdot 2 \\ + \\ 2 \cdot 1 \cdot 1 \end{matrix}}$$

Note: Due to the Cayley-Hamilton theorem, we only have to evaluate $M^{k<N}$ where $N$ is the dimension of $M$.

# Does this mean that I could design my Hamiltonian/matrix to have locally symmetric eigenvectors by tuning its integer-powers?

**Absolutely!**

# Isospectral reductions

- Idea: "Compress" a matrix into a smaller matrix, whilst keeping (almost) all of its spectrum.
- This compression will be of help to understand local symmetries!

# Isospectral reductions

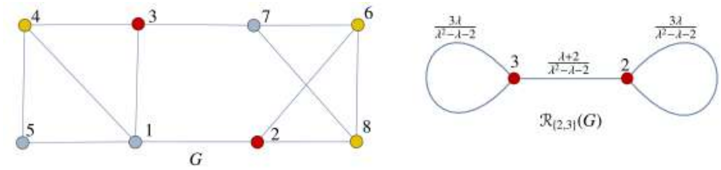



Figure: Taken from[4]

- Given a matrix $M$ (describing a graph), its isospectral reduction $\mathcal{R}_S(M)$ over the set of indices $S$ is defined as

$$\mathcal{R}_S(M,\lambda) = M_{SS} - M_{S\bar{S}}\left(M_{\bar{S}\bar{S}} - \lambda I\right)^{-1} M_{\bar{S}S} \tag{1}$$

and is defined for all values of $\lambda$ that are not eigenvalues of $M_{\bar{S}\bar{S}}$. The entries of $\mathcal{R}_S(M,\lambda)$ are rational functions $\frac{p(\lambda)}{q(\lambda)}$ in $\lambda$.

- The eigenvalues $\lambda_i$ of $\mathcal{R}_S(M,\lambda)$ are the numbers for which

$$det\Big(\mathcal{R}_S(M,\lambda_i) - \lambda_i I\Big) = 0. \tag{2}$$

[4]Smith and Webb, "Hidden Symmetries in Real and Theoretical Networks".

# Eigenvectors of isospectral reductions

- For each eigenvalue $\lambda_i$ of $\mathcal{R}_S(M,\lambda)$, it features one eigenvector $\vec{x_i}$ such that

$$\mathcal{R}_S(M,\lambda_i)\vec{x_i} = \lambda_i \vec{x_i}. \tag{3}$$

  Note that there may be *more* eigenvalues and eigenvectors than the dimension of $\mathcal{R}_S(M,\lambda)$!

- As every eigenvalue $\lambda_i$ of $\mathcal{R}_S(M,\lambda)$ is also an eigenvalue of $M$ (cf. above), there exists an eigenvector $\vec{x'}_i$ of $M$ such that

$$M\vec{x'}_i = \lambda_i \vec{x'}_i. \tag{4}$$

- The eigenvectors of $M$ and $\mathcal{R}_S(M,\lambda)$ are related by

$$(\vec{x'}_i)_S = \vec{x_i} C_i, \tag{5}$$

  i.e., $\vec{x_i}$ is the projection of $\vec{x'}_i$ onto the sites $S$, up to a normalization constant $C_i$.

# Bisymmetric isospectral reductions and latent symmetries

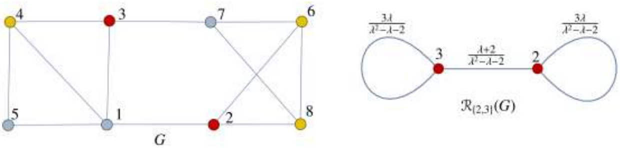



Figure: Taken from[5]

- We now restrict ourselves to the case where $S$ contains only two sites $S = \{u, v\}$ and is *bisymmetric*. An example for such a bisymmetric matrix is

$$\mathcal{R}_{\{2,3\}}(G, \lambda) = \begin{pmatrix} \frac{3\lambda}{\lambda^2-\lambda-2} & \frac{\lambda+2}{\lambda^2-\lambda-2} \\ \frac{\lambda+2}{\lambda^2-\lambda-2} & \frac{3\lambda}{\lambda^2-\lambda-2} \end{pmatrix} = \begin{pmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & a(\lambda) \end{pmatrix}.$$

- For bisymmetric $\mathcal{R}_S(M, \lambda)$, *all* of its eigenvectors are of the form

$$\vec{x}_i = \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \ \forall \ i. \tag{6}$$

[5] Smith and Webb, "Hidden Symmetries in Real and Theoretical Networks".

# Motivation: Perfect and pretty good state transfer

- In any quantum computer, we need to *transfer* qubits throughout the device. Ideally, the qubit of interest would not be affected by the transfer.
- As a simple example, we would model the qubit as a single-site excitation within a spin-network described by

$$H = \frac{1}{2} \sum_{(n,m)\in E} J_n \left( \sigma_n^{(x)} \sigma_m^{(x)} + \sigma_n^{(y)} \sigma_m^{(y)} \right) + \sum_{i=1}^{N} B_i \sigma_i^{(z)} \tag{7}$$

and transfer it across the network by simple time evolution.

# Motivation: Perfect and pretty good state transfer

## Definition

- Given a Hamiltonian $H$, we say that it admits *pretty good state transfer* between $u$ and $v$ if, for any $\epsilon > 0$, there is a time $t > 0$ such that

$$|\langle u|exp(-iHt)|v\rangle| > 1 - \epsilon. \tag{8}$$

where $|u\rangle, (|v\rangle)$ are the vectors with unit amplitude on site $u$ ($v$) and zeros everywhere else.

# Necessary and sufficient conditions for pretty good state transfer

The following are necessary and sufficient conditions for pretty good state transfer on a real-symmetric Hamiltonian $H$ between sites $u$ and $v$.

- Each eigenvector $|x\rangle$ fulfills $\langle u|x\rangle = \pm\langle v|x\rangle$.
- There are no two degenerate eigenvectors $|x_1\rangle, |x_2\rangle$ such that $\langle u|x_1\rangle = +\langle v|x_1\rangle \neq 0,\ \langle u|x_2\rangle = -\langle v|x_2\rangle \neq 0$.
- Let $\{\lambda_i^+\}, \{\lambda_j^-\}$ the eigenvalues associated to eigenvectors of positive/negative parity on $u, v$ and which are non-vanishing on these two sites. Then, any integers $\{l_i, m_j\}$ which fulfill

$$\sum_i l_i \lambda_i^+ + \sum_j m_j \lambda_j^- = 0$$

$$\sum_i l_i + \sum_j m_j = 0$$

  must also fulfill $\sum_i m_i$ is even.

# Designing a system to host pretty good state transfer

## Theorem

*Let*[a] *M be a symmetric matrix with strongly cospectral indices u and v. Further, let*

$$P_{\pm} = \prod_i{}' (\lambda - \lambda_i^{\pm})$$

*where* $\prod'$ *denotes the restriction that degenerate eigenvalues only appear once in the product.*
*Then, if*

- $P_+, P_-$ *are irreducible and have no common root, and if*
- $\frac{Tr(P_+)}{deg(P_+)} \neq \frac{Tr(P_-)}{deg(P_-)}$ *where Tr denotes the sum of roots of a polynomial,*

*then there is pretty good state transfer between u and v.*

[a] Theorem 2.11. from Eisenberg, Kempton, and Lippner, "Pretty Good Quantum State Transfer in Asymmetric Graphs via Potential"

# The connection between isospectral reductions and $P_{\pm}$

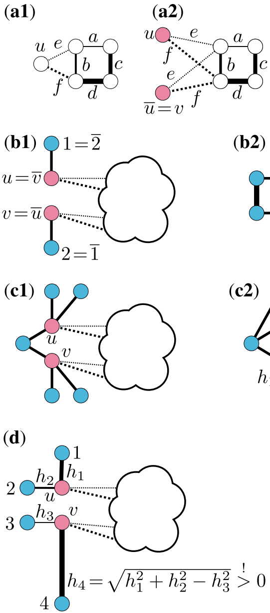
- Let $\mathcal{R}_S(M,\lambda) = \begin{pmatrix} a(\lambda) & b(\lambda) \\ b(\lambda) & a(\lambda) \end{pmatrix}$ bisymmetric.
- We then perform the similarity transform

$$\mathcal{R}'_S(M,\lambda) = A^{-1}\mathcal{R}_S(M,\lambda)A = \begin{pmatrix} a(\lambda)+b(\lambda) & 0 \\ 0 & a(\lambda)-b(\lambda) \end{pmatrix}$$

with $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. We further define $P'_{\pm}(\lambda) = a(\lambda_i) \pm b(\lambda_i) - \lambda_i$.

- The eigenvalues $\lambda_i$ of $\mathcal{R}'_S(M,\lambda)$ and $\mathcal{R}_S(M,\lambda)$ are identical and given by the roots of $P'_{\pm}(\lambda)$, respectively.
- The eigenvectors of $\mathcal{R}_S(M,\lambda)$ are given by multiplying the corresponding eigenvectors of $\mathcal{R}'_S(M,\lambda)$ by $A$. Therefore, these are obviously $(1,1)^T$ [for eigenvalues $\lambda_i$ fulfilling $P'_{+}(\lambda_i) = 0$] and $(1,-1)^T$ [for eigenvalues $\lambda_i$ fulfilling $P'_{-}(\lambda_i) = 0$].
- The roots of the polynomials $P'_{\pm}(\lambda)$ therefore are the eigenvalues of the locally symmetric/anti-symmetric eigenvectors (which can be chosen not to vanish on $u, v$) of $M$, respectively!

# Designing cospectral graphs



# How to design graphs with pretty good state transfer

In principle (details omitted here), pretty good state transfer can be obtained by the following algorithm:

1. Design a parameter-dependent setup $H(\xi)$ with cospectral vertices $u, v$.
2. Tune its eigenvalue spectrum to be non-degenerate.
3. Obtain the polynomials $P_{\pm}(\xi)$.
4. Properly tune $\xi$ such that $P_{\pm}(\xi)$ are irreducible and fulfill $\frac{Tr(P_+)}{deg(P_+)} \neq \frac{Tr(P_-)}{deg(P_-)}$. If there are no eligible $\xi$, go back to step 1., change the Hamiltonian (e.g., by adding sites) and start the algorithm anew.

# Open questions

- Can we generalize the concept of cospectrality to more than two vertices?
- Can we do a block-diagonalization of the Hamiltonian if there are cospectral vertices? For cospectrality induced by global symmetries, it is possible.
- What can we learn about "our" local symmetries from applying the isospectral reduction?

# Thank you!

# References I


Eisenberg, Or, Mark Kempton, and Gabor Lippner. "Pretty Good Quantum State Transfer in Asymmetric Graphs via Potential". In: *Discrete Math.* (Nov. 28, 2018). ISSN: 0012-365X. DOI: 10.1016/j.disc.2018.10.037. URL: http://www.sciencedirect.com/science/article/pii/S0012365X18303716 (visited on 07/01/2019).

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# References II

Zheng, Li-Yang et al. "Observation of Edge Waves in a Two-Dimensional Su-Schrieffer-Heeger Acoustic Network". In: (Mar. 26, 2019). arXiv: 1903.11961 [physics]. URL: http://arxiv.org/abs/1903.11961 (visited on 08/25/2019).