Disorder analysis in 1-D topological lattices

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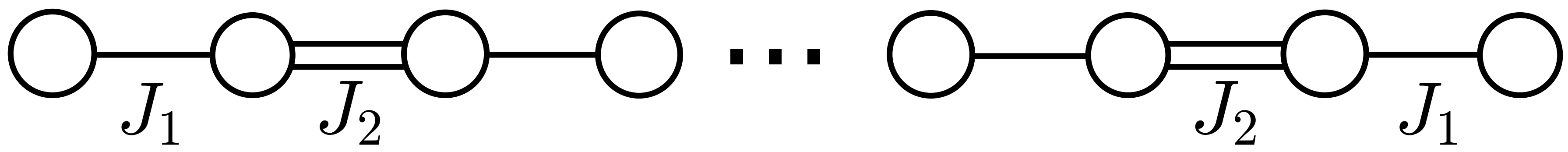
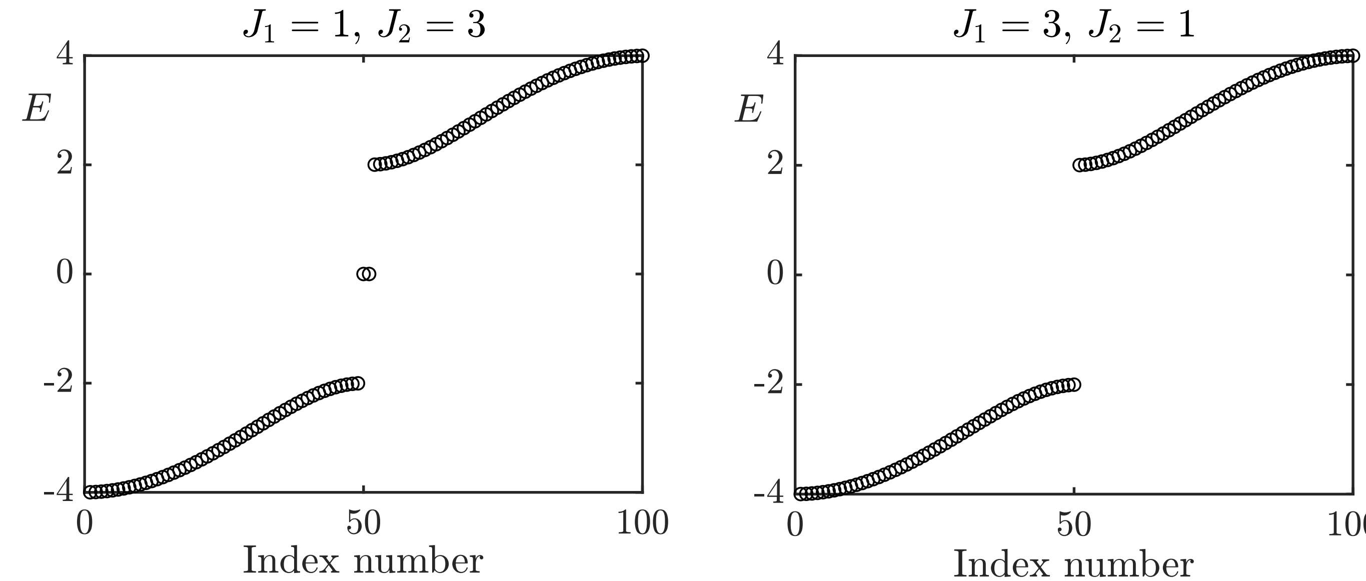
Local symmetries in wave physics
September 2019 , Karystos

SSH model



$$\mathcal{H} = \begin{pmatrix} 0 & J_1 & 0 & 0 & \cdots & 0 \\ J_1 & 0 & J_2 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & J_1 & \cdots & 0 \\ & \ddots & & \ddots & & \\ 0 & \cdots & 0 & J_2 & 0 & J_1 \\ 0 & \cdots & 0 & 0 & J_1 & 0 \end{pmatrix}$$



↳ Chiral symmetry ?

A system possesses chiral symmetry if $\hat{\Sigma}\hat{\mathcal{H}}\hat{\Sigma} = -\hat{\mathcal{H}}$

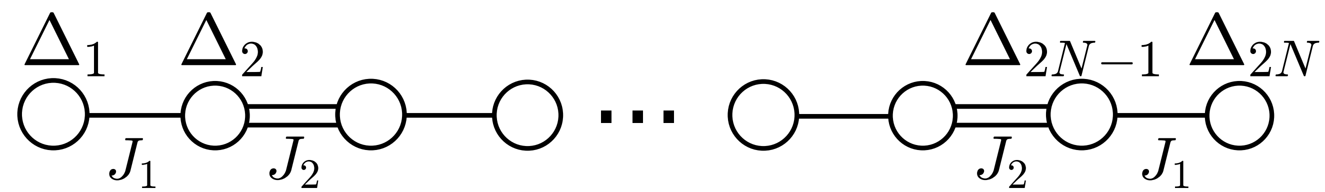
$\hat{\Sigma}$ has to be unitary and Hermitian
it is also a local operator

For the SSH

$$\hat{\Sigma} = \hat{\sigma}_z \oplus \hat{\sigma}_z \oplus ... \oplus \hat{\sigma}_z$$

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & \cdots & 0 \\ & \ddots & & \ddots & & \\ 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 \end{pmatrix}$$

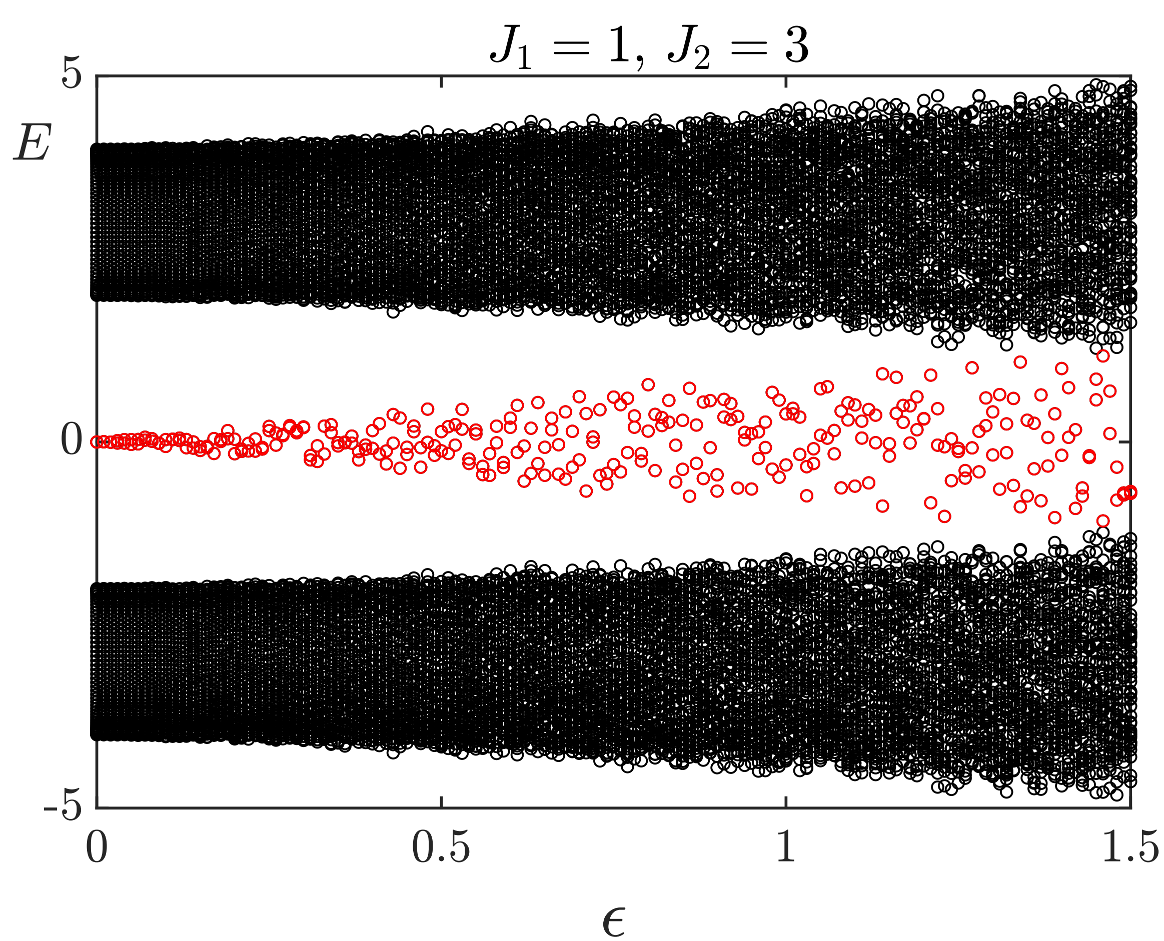
Disorder to the on-site potential



$$\mathcal{H} = \begin{pmatrix} \Delta_1 & J_1 & 0 & 0 & \cdots & 0 \\ J_1 & \Delta_2 & J_2 & 0 & \cdots & 0 \\ 0 & J_2 & \Delta_3 & J_1 & \cdots & 0 \\ 0 & 0 & J_1 & \Delta_4 & \cdots & 0 \\ & \ddots & & \ddots & & \\ 0 & \cdots & 0 & J_2 & \Delta_{2N-1} & J_1 \\ 0 & \cdots & 0 & 0 & J_1 & \Delta_{2N} \end{pmatrix}$$

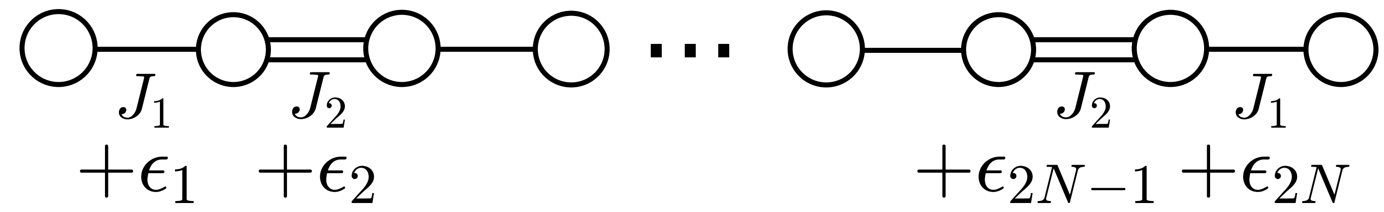
This Hamiltonian doesn't possess chiral symmetry

$$\hat{\Sigma}\hat{\mathcal{H}}\hat{\Sigma} \neq -\hat{\mathcal{H}}$$



- The edge modes are not robust
- The two bands are not symmetric with respect to zero

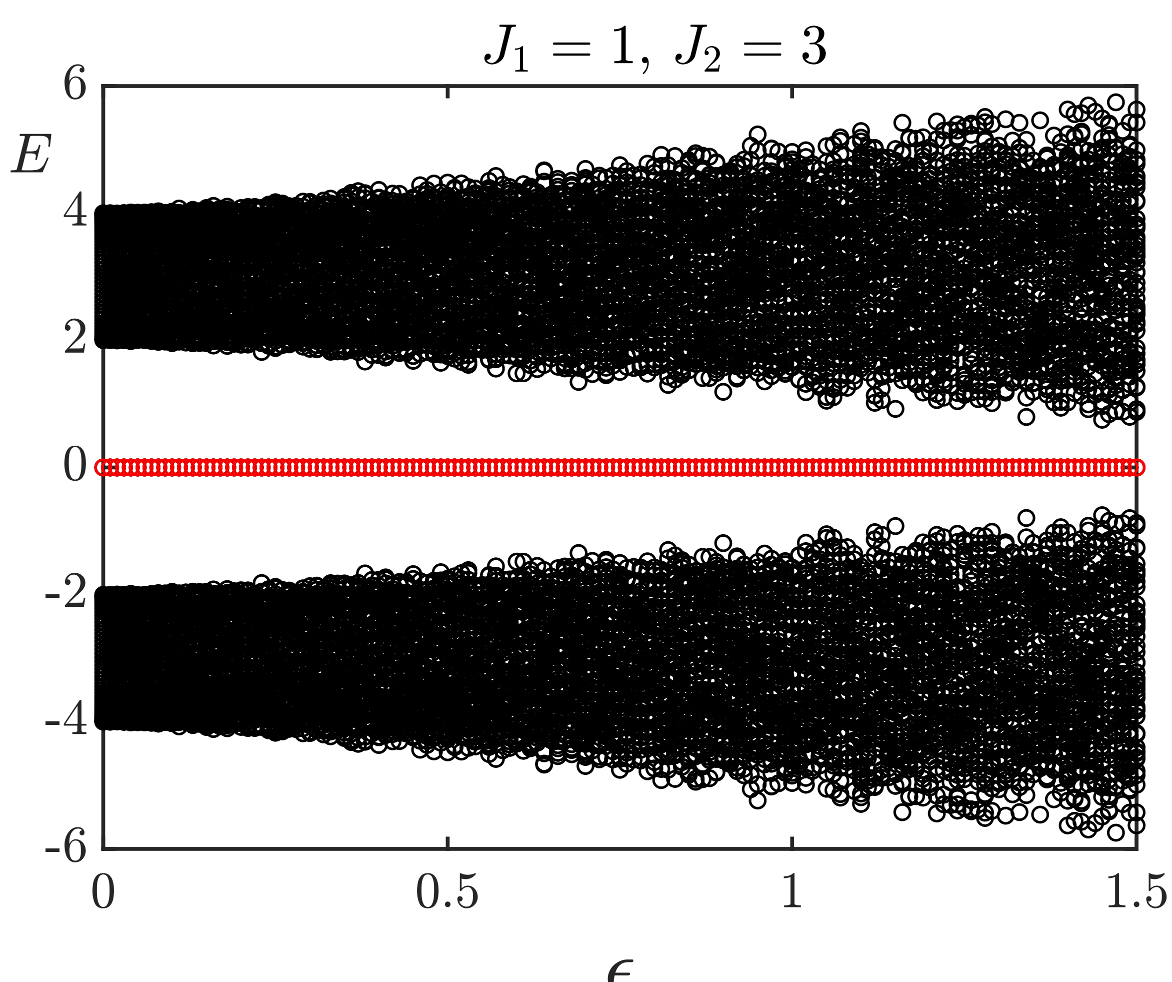
Disorder to the couplings



$$\mathcal{H} = \begin{pmatrix} 0 & J_1+\epsilon_1 & 0 & 0 & \cdots & 0 \\ J_1+\epsilon_1 & 0 & J_2+\epsilon_2 & 0 & \cdots & 0 \\ 0 & J_2+\epsilon_2 & 0 & J_1+\epsilon_3 & \cdots & 0 \\ 0 & 0 & J_1+\epsilon_3 & 0 & \cdots & 0 \\ & \ddots & & \ddots & & \\ 0 & \cdots & 0 & J_2+\epsilon_{2N-1} & 0 & J_1+\epsilon_{2N} \\ 0 & \cdots & 0 & 0 & J_1+\epsilon_{2N} & 0 \end{pmatrix}$$

This Hamiltonian still possesses chiral symmetry

$$\hat{\Sigma}\hat{\mathcal{H}}\hat{\Sigma} = -\hat{\mathcal{H}}$$



- The edge modes are robust
- The two bands are symmetric with respect to zero

Localization length λ of zero energy mode

$$M_1 = \begin{pmatrix} \frac{E}{J_2} & -\frac{J_1}{J_2} \\ 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} \frac{E}{J_1} & -\frac{J_2}{J_1} \\ 1 & 0 \end{pmatrix}$$

$M_1,\ M_2$: Transfer matrices

E : Energy

Adding disorder to the couplings
$$J_{1,n} = 1 + W_1 \omega_n$$
$$J_{2,n} = m + W_2 \omega'_n$$

W_1, W_2 : Disorder strengths

ω_n, ω'_n : Random numbers, uniformly distributed in the interval [-0.5,0.5]

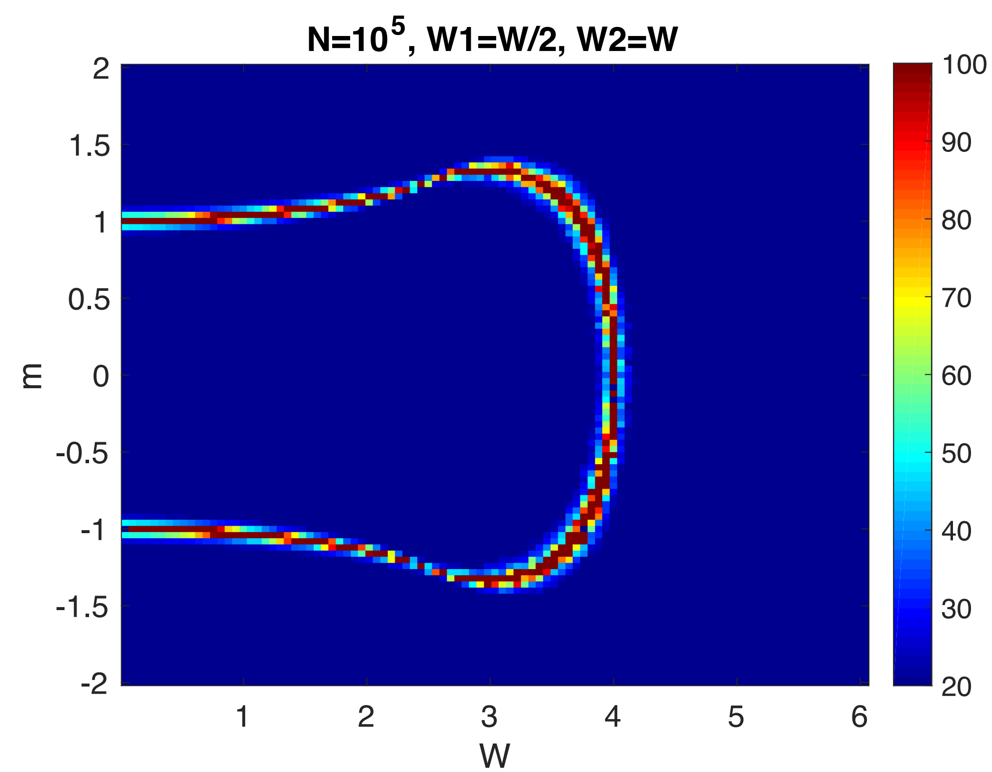
We calculate the Lyapunov exponent γ However, $\gamma \sim 1/\lambda$

$$E = 0$$

$\downarrow$

$$M_1 = \begin{pmatrix} 0 & -\frac{J_1}{J_2} \\ 1 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & -\frac{J_2}{J_1} \\ 1 & 0 \end{pmatrix}$$



I. Mondragon-Shem, et. al.
Phys. Rev. Lett. **113,** 046802 (2014).

2x2 transfer matrix problem $\begin{pmatrix} A_{\nu+1} \\ A_\nu \end{pmatrix} = \hat{M}_\nu \begin{pmatrix} A_\nu \\ A_{\nu-1} \end{pmatrix}$ or $\begin{pmatrix} A_{\nu+1} \\ A_\nu \end{pmatrix} = \hat{T}_\nu \begin{pmatrix} A_1 \\ A_0 \end{pmatrix} \Rightarrow \vec{A}_\nu = \hat{T}_\nu \vec{A}_0$

where $\hat{T}_\nu = \prod_{n=1}^{\nu} \hat{M}_n$

Furstenberg 1963 Maximum Lyapunov exponent

$$\gamma_1 = \lim_{N\to\infty} \frac{1}{N} \ln ||\hat{T}_N \vec{z}||$$

$\vec{z}$ is a random vector

$$\Downarrow$$

$$\gamma_1 = \lim_{N\to\infty} \frac{1}{2N} \ln(A_{N+1}^2 + A_N^2)$$

$$\gamma_1 = \frac{1}{N} \ln(||\vec{A}_N||) = \frac{1}{N} \ln \left(\frac{||\vec{A}_N||}{||\vec{A}_0||} \right)$$

if $||\vec{A}_0|| = 1$

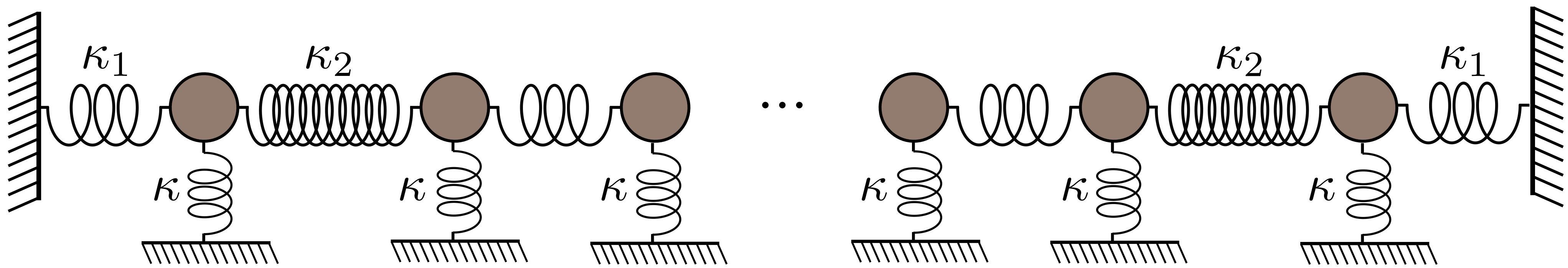
$$\Downarrow$$

$$\gamma_1 = \frac{1}{N} \ln \left(\frac{||\vec{A}_N||}{||\vec{A}_{N-1}||} \frac{||\vec{A}_{N-1}||}{||\vec{A}_{N-2}||} \cdots \frac{||\vec{A}_1||}{||\vec{A}_0||} \right)$$

$\gamma_1 = \frac{1}{N} S_N = \frac{1}{N} \sum_{i=1}^{N} S_i'$ where $S_i' = \ln \left(\frac{||\vec{A}_i||}{||\vec{A}_{i-1}||} \right)$

Normalizing the vectors in the previous step $S_{N+1} = S_N + \ln \left(||\vec{A}_{N+1}|| \right)$

Dimer lattice



$u_i, \, \forall 1 \leq i \leq N$ displacements

Assuming that all masses oscillate with the same frequency ω we can write $u_i = v_i e^{-i\omega t}$

$$\Rightarrow -\omega^2 \hat{M}|U\rangle = \hat{K}|U\rangle \quad \text{where} \quad |U\rangle = \begin{pmatrix} v_1 \\ v_2 \\ \\ \vdots \\ \\ v_N \end{pmatrix}, \quad \hat{M} = \begin{pmatrix} m & 0 & 0 & \cdots & 0 \\ 0 & m & 0 & \cdots & 0 \\ & & \ddots & & \\ 0 & \cdots & 0 & m & 0 \\ 0 & \cdots & 0 & 0 & m \end{pmatrix}$$

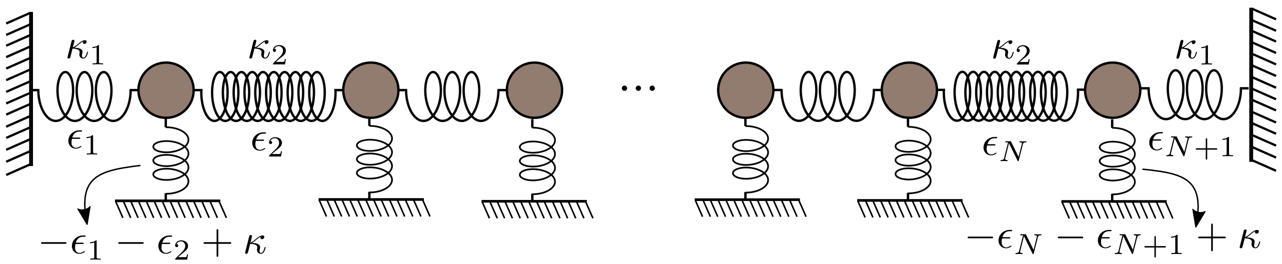
and

$$\hat{K} = \begin{pmatrix} -\kappa_1-\kappa_2-\kappa & \kappa_2 & 0 & 0 & \cdots & & \cdots & 0 \\ \kappa_2 & -\kappa_1-\kappa_2-\kappa & \kappa_1 & 0 & \cdots & & \cdots & 0 \\ 0 & \kappa_1 & -\kappa_1-\kappa_2-\kappa & \kappa_2 & 0 & & \cdots & 0 \\ \vdots & & \ddots & & & & & \vdots \\ 0 & \cdots & \kappa_2 & -\kappa_1-\kappa_2-\kappa & \kappa_1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \kappa_1 & -\kappa_1-\kappa_2-\kappa & \kappa_2 & \cdots & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & \cdots & & 0 & \kappa_2 & -\kappa_1-\kappa_2-\kappa & \kappa_1 & 0 \\ 0 & \cdots & & \cdots & 0 & \kappa_1 & -\kappa_1-\kappa_2-\kappa & \kappa_2 \\ 0 & \cdots & & \cdots & 0 & 0 & \kappa_2 & -\kappa_1-\kappa_2-\kappa \end{pmatrix}$$

The matrix $\hat{K}$ "anti-commutes" with the matrix

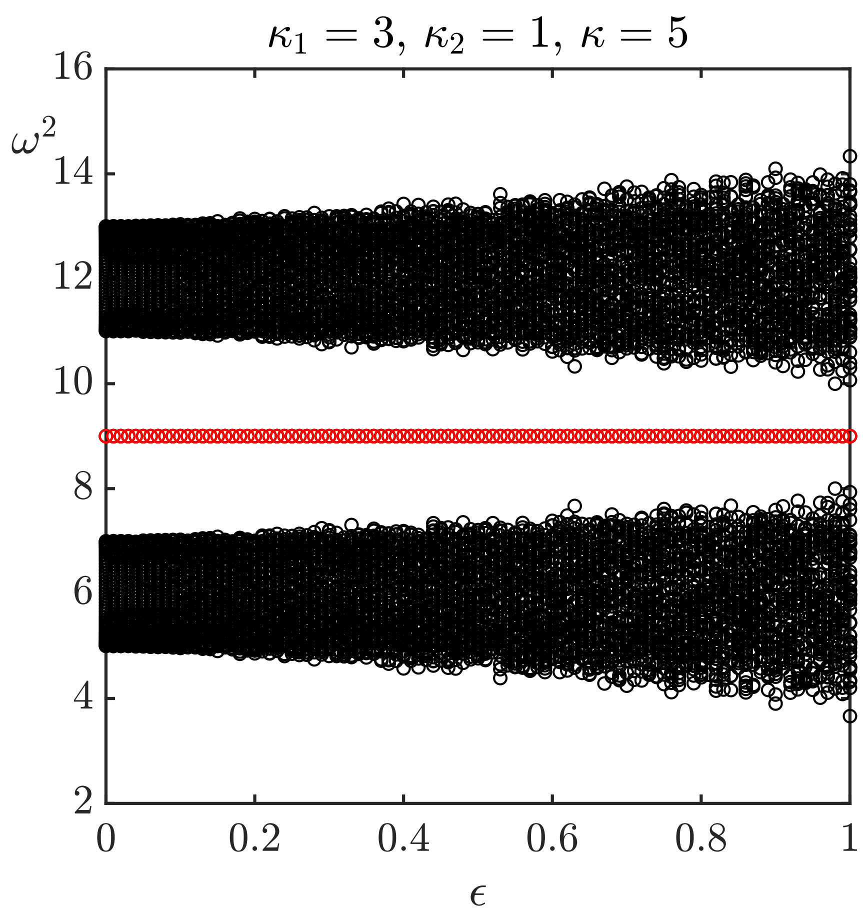
$\hat{\Sigma}_z = \hat{\sigma}_z \oplus \hat{\sigma}_z \oplus \cdots \oplus \hat{\sigma}_z \oplus \hat{\sigma}_z$ **in the sense** $\{\hat{K}, \hat{\Sigma}_z\} = -2\left(\kappa_1 + \kappa_2 + \kappa\right)$

Disorder respecting chiral symmetry

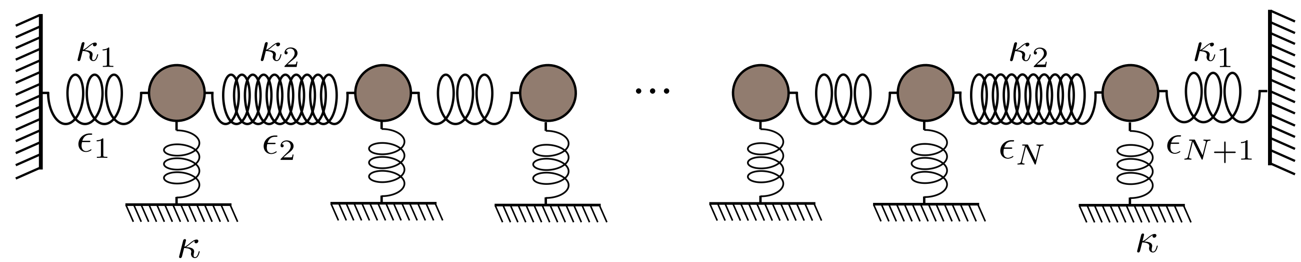


The diagonal term is not affected by the disorder

$$= -\kappa_1 - \kappa_2 - \kappa$$

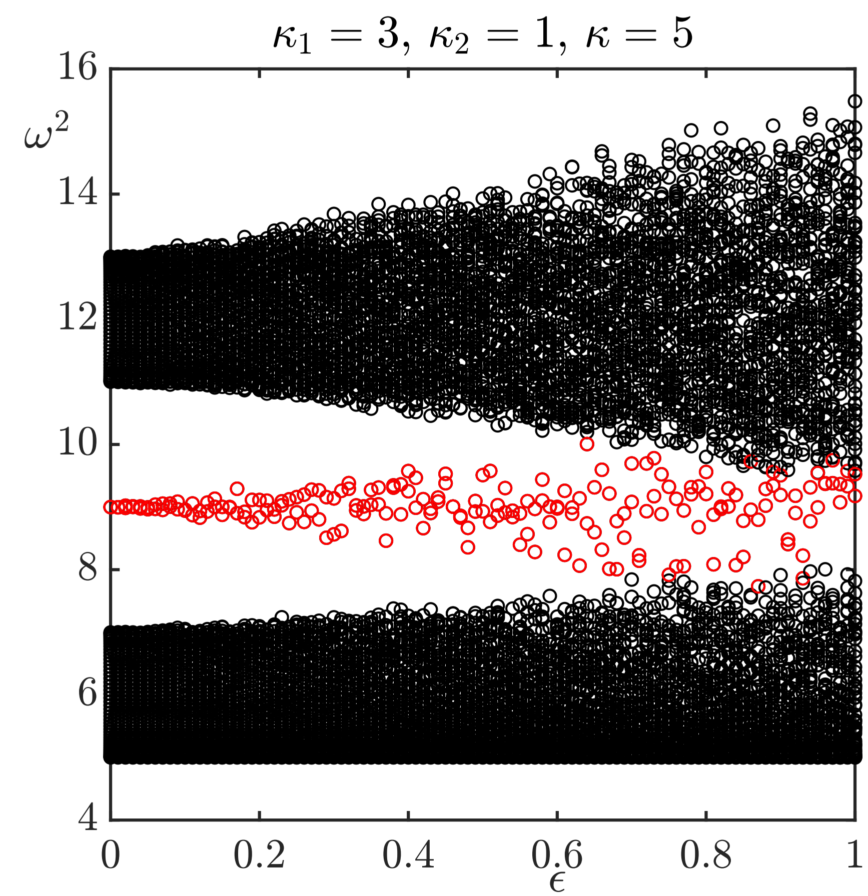


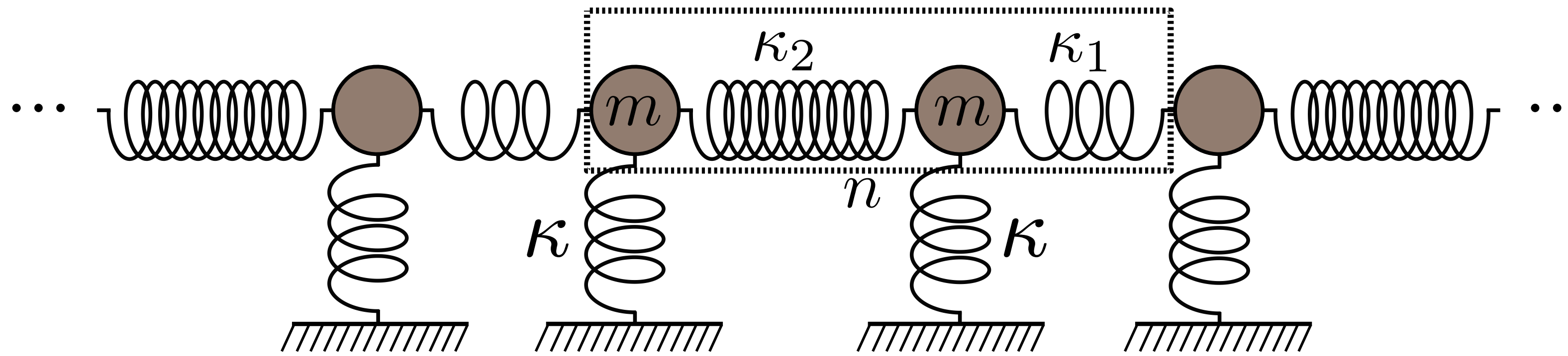
Disorder not respecting chiral symmetry



The diagonal term is affected by the disorder

$$\neq -\kappa_1 - \kappa_2 - \kappa$$





$$M_1 = \begin{pmatrix} 1 + \frac{\kappa_1}{\kappa_2} + \kappa - \frac{m\omega^2}{\kappa_2} & -\frac{\kappa_1}{\kappa_2} \\ 1 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 1 + \frac{\kappa_2}{\kappa_1} + \kappa - \frac{m\omega^2}{\kappa_1} & -\frac{\kappa_2}{\kappa_1} \\ 1 & 0 \end{pmatrix}$$

Frequency of the edge mode

$$m\omega^2 = \frac{\kappa_1 + \kappa_2 + \kappa}{m}$$

↓

$$M_1 = \begin{pmatrix} 0 & -\frac{\kappa_1}{\kappa_2} \\ 1 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & -\frac{\kappa_2}{\kappa_1} \\ 1 & 0 \end{pmatrix}$$

Adding disorder to the couplings

$$\kappa_1 = 1 + W_1\omega_n$$
$$\kappa_2 = m + W_2\omega_n'$$

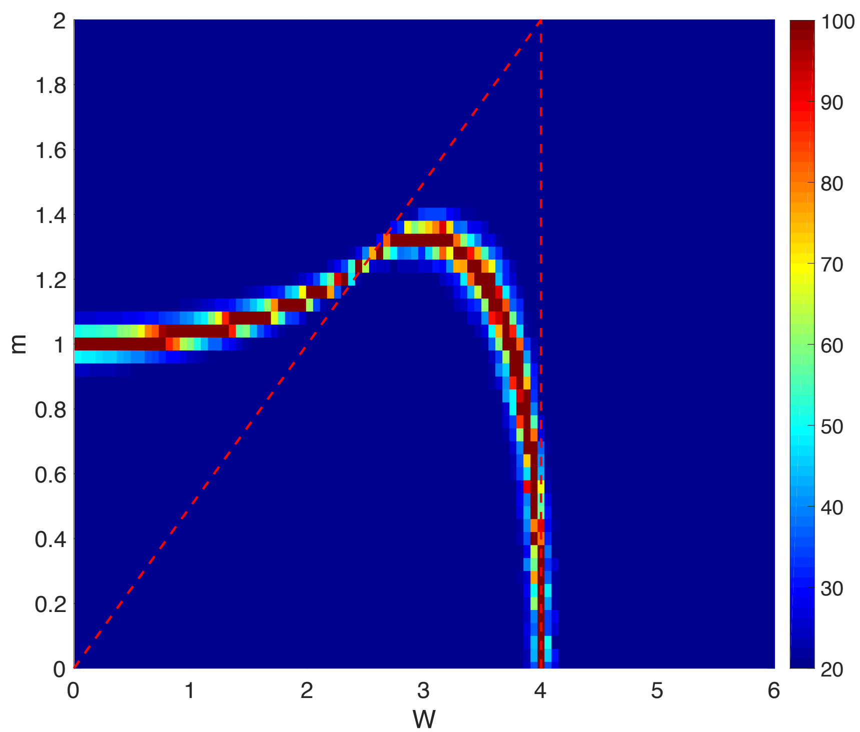
The spring constants cannot take negative values ⟶

$$W_1 \leq 2$$
$$W_2 \leq 2m$$

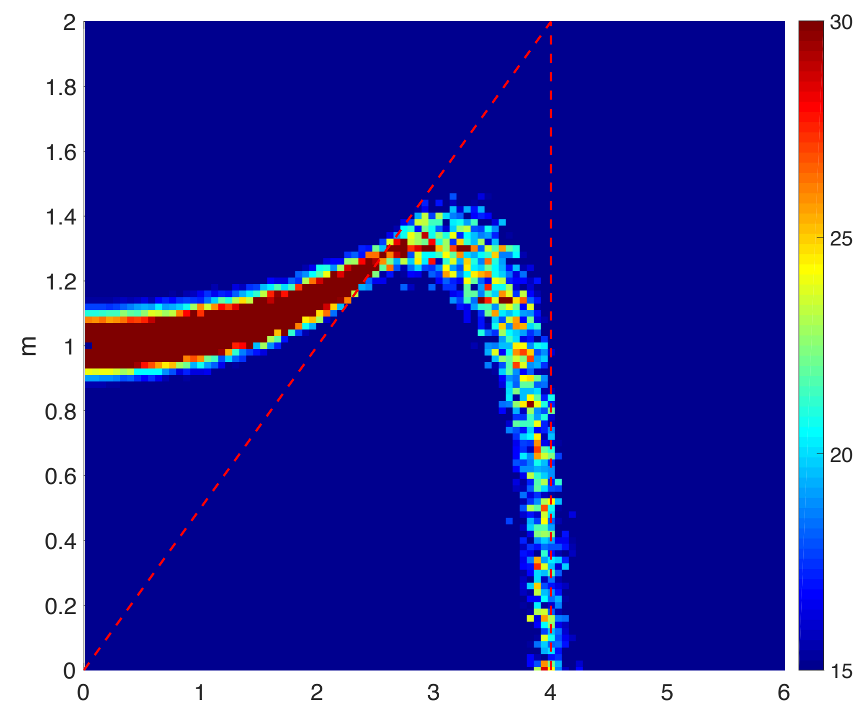
For disorder strengths equal to

$$W_2 = W \qquad W_1 = W/2 \quad \longrightarrow \quad W \leq 4 \qquad W \leq 2m$$

SSH



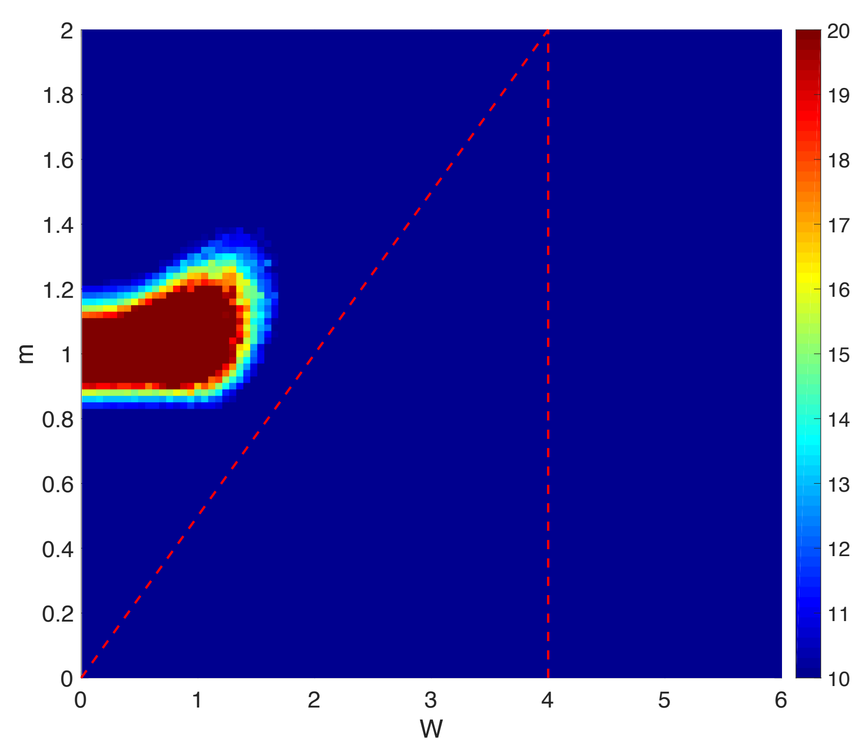
Phononic with disorder respecting chiral symmetry



The frequency is kept constant

$$m\omega^2 = \frac{\kappa_1 + \kappa_2 + \kappa}{m}$$

Phononic with random disorder



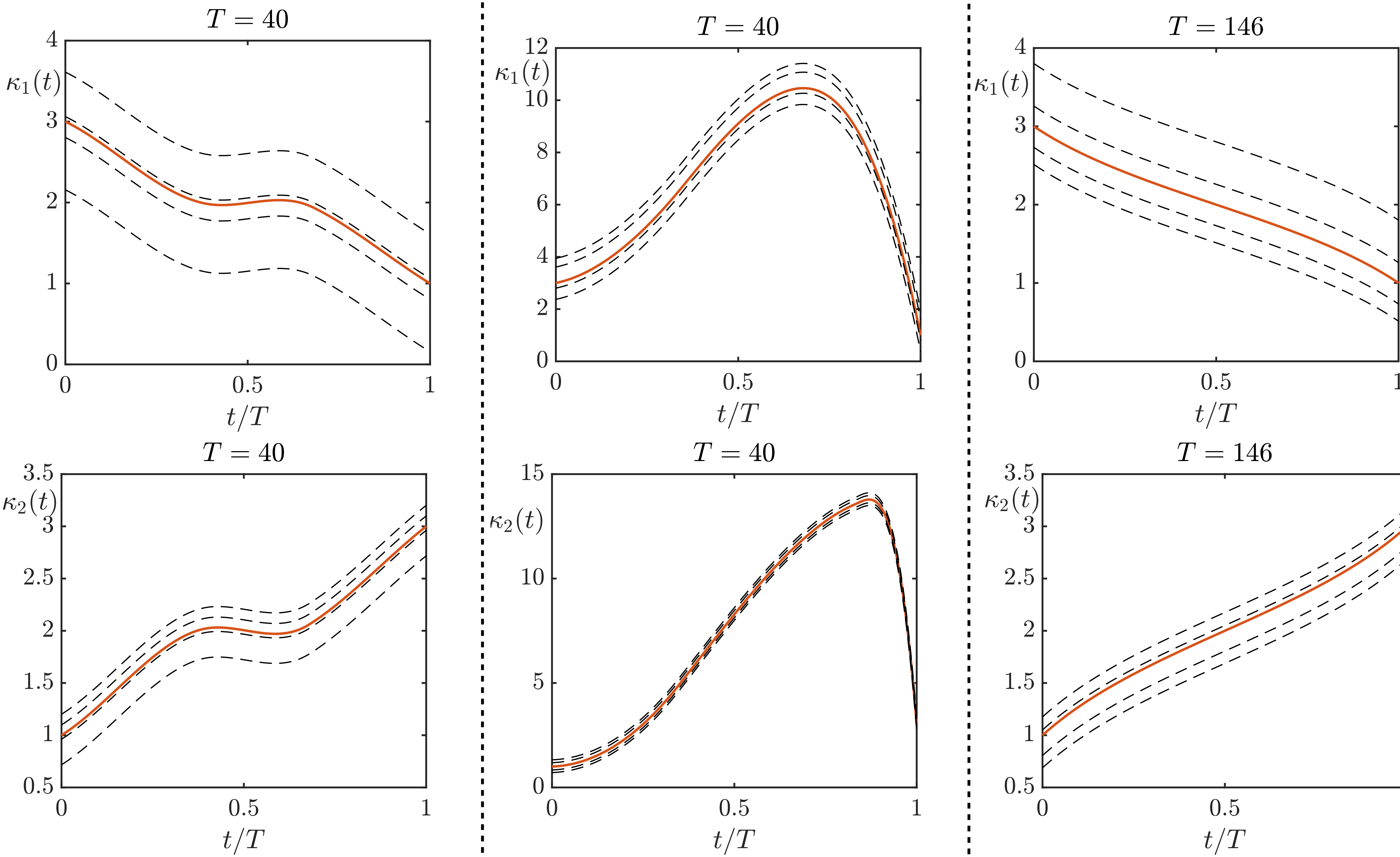
The frequency is not kept constant

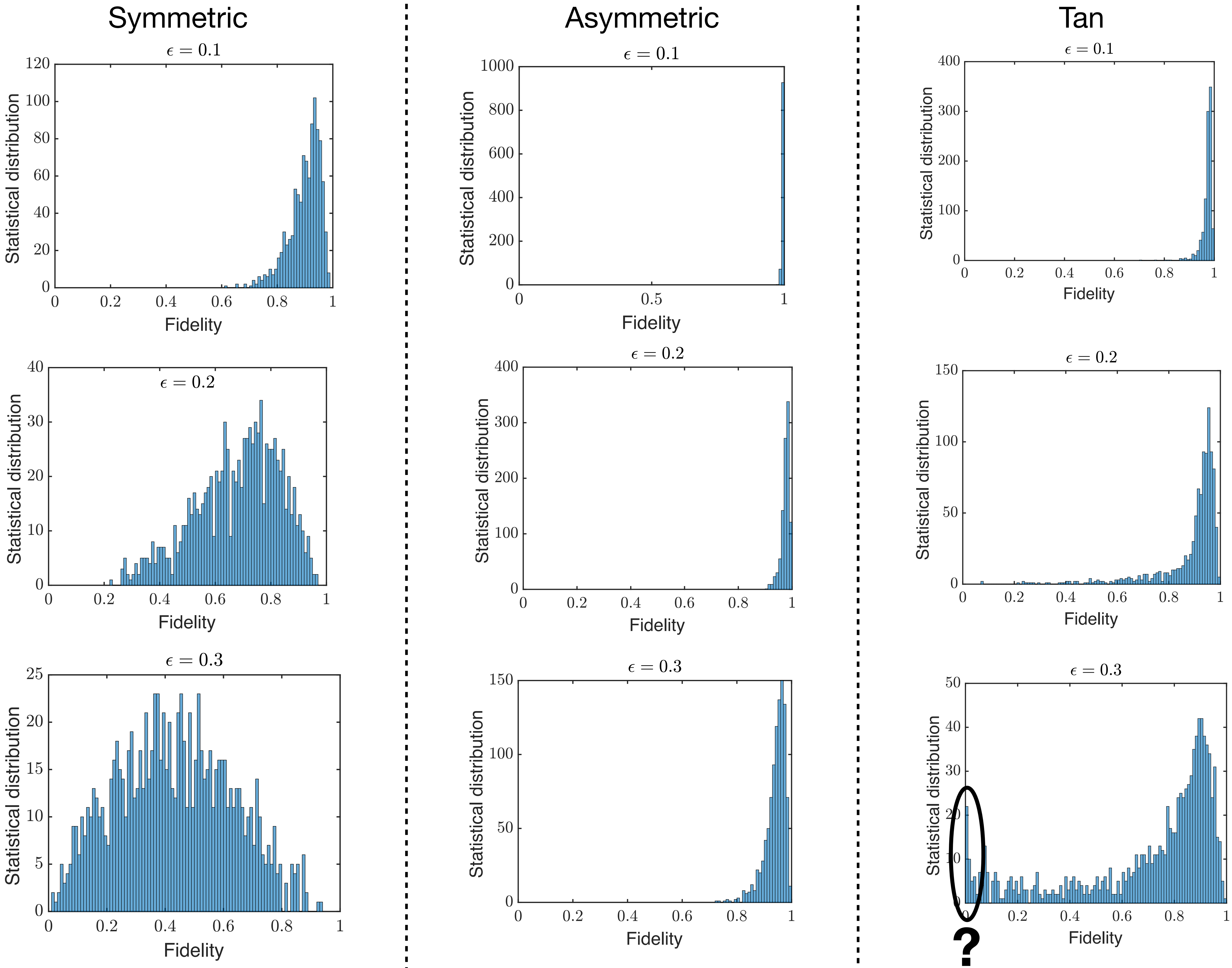
Time dependent springs
Optimization

$$\kappa_{1,2}(0) \to \kappa_{1,2}(0)\,(1 + \epsilon r)$$

r: Random number, uniformly distributed in the interval [-1,1]

Symmetric | Asymmetric | Tan




Symmetric
Asymmetric
Tan
$\epsilon = 0.1$
$\epsilon = 0.2$
$\epsilon = 0.3$
Statistical distribution
Fidelity
?

Thank you!